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CONSOLIDATION THEORY DERIVED WITHOUT INVOKING POROSITY

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Abstract—Biot and Willis wrote that consolidation theory could be derived without reference to porosity. I do so in this article. The coefficients in the consolidation equation that results are easier to understand than Biot's. In special cases the theory leads to three-dimensional diffusion equations. One such is when all constituents of the material are intrinsically incompressible, a good approximation for articular cartilage. In one dimension the conditions leading to a diffusion equation are much broader. Published by Elsevier Science Ltd.

NOTATION

Symbols that have the same meaning here as in Biot (1955) or Biot and Willis (1957) are identified. Symbols found in the final consolidation equations in this paper are marked with a heart ♥.

$B = (\partial p/\partial e)_{\zeta=\text{const.}}$ $= [\partial(\tau'-p)/\partial\zeta]_{e=\text{const.}}$ ♥	The rise in pore pressure caused by an increase e in extent in the absence of flow into or out of the sample. Also, the rise in tensile total stress $\tau'-p$ caused by an injection of fluid into the sample at constant extent. B is negative.
$e = e_{xx}+e_{yy}+e_{zz}$ ♥	Relative increase in the extent of the solid skeleton, i.e. expansion of the volume throughout which the solid structure extends divided by the volume (Biot (1955), Biot and Willis (1957)).
e_{ij}	Deformation, the slope of displacement in the i direction with distance in the j direction (Biot (1955), Biot and Willis (1957)).
e_{xx}, e_{yy}, e_{zz}	Relative expansion of the solid structure in the x, y and z directions, respectively (Biot (1955), Biot and Willis (1957)).
K ♥	Flow resistivity of the material.
$M = (\partial p/\partial\zeta)_{e=\text{const.}}$ ♥	Rise in pore pressure per unit volume of fluid injected into a unit volume of material when the volume throughout which the solid extends, the extent, is kept constant. (Much less than unit volume would be injected in an actual test.) The solid material must be sufficiently pre-compressed that it does not pull away from the walls of the container during the test. Alternatively it can be attached to the walls. M is positive (Biot and Willis (1957)).
p ♥	Pore pressure (Biot and Willis (1957)).
t ♥	Time (Biot (1955)).
x	A particular axis.
$Z = [\partial(\tau'-p)/\partial e]_{\zeta=\text{const.}}$	Bulk modulus of the whole material in the absence of redistribution of fluid. The material is compressed in a jacket that prevents fluid from entering or leaving it. If necessary the solid of the material is precompressed sufficiently to keep it from pulling away from the jacket. Alternatively it can be attached to the jacket. Z is positive.
$\delta = -(\partial e/\partial p)_{\tau'=\text{const.}}$ ♥	Pore pressure compressibility of the skeleton domain, the negative of the proportional increase in extent (which is negative) per unit rise in pore pressure at constant effective stress. If the solid forms a defined phase, δ is the compressibility of the solid (Biot and Willis (1957)).
$\delta/\kappa = (\partial\tau'/\partial p)_{e=\text{const.}}$ $= [\partial(e-\zeta)/\partial e]_{p=\text{const.}}$ ♥	The increase in effective stress caused by a unit increase in pore pressure at constant extent; also the difference between the increase in extent at constant pressure and the fluid inflow (Biot and Willis (1957)).
$1-\delta/\kappa = \alpha = (\partial\zeta/\partial e)_{p=\text{const.}}$ $= -[\partial(\tau'-p)/\partial p]_{\zeta=\text{const.}}$ ♥	The volume of fluid that flows into a sample whose extent has been increased by one unit volume at constant pore pressure; also the negative of the rise in total tension with increase in pore pressure if flow of fluid is prevented. (Biot and Willis (1957)).
$\gamma = (\partial\zeta/\partial p)_{\tau'=\text{const.}}$	The coefficient of fluid content, the amount of fluid that enters a unit volume of unrestrained sample per unit increase in pore pressure. The boundaries across which the flow is defined are fixed to the skeleton and shrink along with the sample as the pore pressure is increased (Biot and Willis (1957)).
$\kappa = (\partial e/\partial\tau')_{p=\text{const.}}$ ♥	Effective-stress compressibility of the skeleton domain, the proportional expansion of the extent of the solid per unit isotropic effective tensile stress at constant pore pressure (Biot and Willis (1957)).

λ	Lamé constant of the skeleton structure, i.e., the Lamé constant of the entire material at constant pore pressure (Biot and Willis (1957)).
μ	Shear modulus of the skeleton structure. This is also the shear modulus of the entire material. It is independent of pore pressure (Biot and Willis (1957)).
v	Poisson's ratio of the skeleton structure, i.e., of the whole material at constant pressure.
τ_{ij}	The component of total stress in the i direction across a plane normal to the j axis (Biot and Willis (1957)).
$\tau' = (\tau'_{xx}+\tau'_{yy}+\tau'_{zz})/3$	Average effective tensile stress.
τ'_{ij}	The component of effective stress in the i direction across a plane normal to the j axis. When $i = j$, $\tau'_{ij} = \tau'_{ii} = \tau_{ii}+p$. When $i \neq j$, $\tau'_{ij} = \tau_{ij}$ (Biot and Willis (1957)).
$\Xi = 2\mu+\lambda$♥	The confined compression modulus of the material at constant pore pressure. Also called the aggregate modulus.
ζ	Fluid content, the volume of fluid in material initially of unit volume in excess of that when p and τ' are both zero. The boundary of the test volume is fixed to the skeleton of the material (Biot and Willis (1957)).

INTRODUCTION

Familiar treatments of poroelasticity such as those in Biot (1941, 1955), Biot and Willis (1957), Kenyon (1976), Mow and Lai (1980), Mow *et al.* (1980), and Simon (1992) use a stress separation system. They divide the total stress in the poroelastic material into stress in the solid and stress in the fluid. Fluid stress is the negative of the pore pressure (tension is taken as positive) times the porosity. The porosity is the volume occupied by fluid divided by the total volume of the material. Subtracting fluid stress from total stress leaves the solid stress.

Soil mechanicians, in contrast, use a stress-superposition system. Pore pressure is taken to exist throughout the material, solid as well as fluid. Physically this is true, because pore pressure causes an equal isotropic compressive stress in the solid. Subtracting pore pressure from the total compressive stress leaves "effective compressive stress", the consequence of those stresses in the solid that are additional to the hydrostatic stress.

Total stress and pore pressure are quantities of direct mechanical significance, the ones an engineer building a dam would measure or want to predict. To re-express these stresses in the stress-separation system one must assume a porosity, but any value can be used because it drops out when the stresses are converted back into total stress and pore pressure.

Porosity *per se* has no effect on consolidation. It is the flow permeability consequent on the presence of the pores that is significant. Obviously a sponge of intrinsically incompressible material that has 10% porosity cannot lose more than 10% of its volume, but the loss in pore volume affects consolidation only via reduction in the permeability of the sponge structure and increase in its stiffness moduli.

Harrigan and Mann (1987) noted that poroelastic behavior does not require that the material be comprised of distinct solid and fluid phases. Pore pressure at a particular point is defined by Harrigan and Mann as the pressure in pure fluid in equilibrium with the fluid at the point. Imagine that a tube full of pure fluids leads from the point to a pressure guage.

Because the explicit presence of porosity in consolidation theory is a detour that has no effect on the results it should be possible to leave it out. Biot and Willis (1957, pp. 597–598) agree. Having introduced the stress-superposition system they say, "Furthermore, if the Darcy equation for volume flow is used, all the equations of consolidation theory may be developed without reference to porosity." They did not give the derivation.

I give it in this article, in part because the presence of porosity has led to confusion about the matching conditions at a boundary between poroelastic material and fluid. For example Mansour and Mow (1977, p. 166) wrote that "surface area porosity" (the porosity right at the surface as distinct from the porosity in the bulk of the material) affects the apportioning of stress within the material when the latter is faced with pressurized fluid.

DERIVATION

If fluid and solid do not form distinct phases in the material, pore pressure at a particular place in the sample is taken in the Harrigan and Mann (1987) sense, i.e., the pressure in pure fluid in equilibrium with that in the material at that place.

Without appeal to strain energy the present derivation yields a consolidation equation in terms of four mechanical constants and the permeability. Because the material is assumed to have a single-valued potential energy of strain, two of the constants are identical.

Biot (1955) described strain energy in terms of the strains of solid and fluid, and found the various stresses by differentiating this expression by the strains that the stresses would cause. The coefficients of the terms of the energy formula appear in these derivatives. They are unfamiliar, and Biot and Willis (1957) spent many difficult paragraphs explaining how they are related to mechanical moduli that can be measured. My way of describing the state of the material leads to an expression for strain energy in terms of directly measurable moduli.

The conventions and symbols of Biot and Willis (1957) will be followed as far as possible.

Accelerations of fluid and solid are so small in consolidation phenomena that inertial reactions are negligible. The total force and total torque on any volume of material are thus taken to be zero. I will show that this makes the Laplacian of dilatation proportional to the Laplacian of pore pressure, which causes a proportional local influx of fluid and a consequent rise with time of the dilatation, the pore pressure, or both. That, in words, is eqn (22), the run-on (i.e., with two equals signs) equation of consolidation. Equations (26), (27) and (41) are eqn (22) after one of its coefficients has been eliminated by the appeal to strain energy.

The state of stress in any tiny volume of an isotropic material is given by the nine element matrix

$$\begin{bmatrix} \tau_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \tau_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \tau_{zz} \end{bmatrix}. \tag{1}$$

The strain tensor of the solid skeleton is

$$\begin{bmatrix} e_{xx} & e_{xy} & e_{xz} \\ e_{yx} & e_{yy} & e_{yz} \\ e_{zx} & e_{zy} & e_{zz} \end{bmatrix}. \tag{2}$$

The e_{ij} are deformations of the structure, not of the individual rods, strips or sheets that make it up. As an analogy at a larger scale, think of the bends and loss of height of an earthquake-damaged building, not the twisting and buckling of its individual pillars, walls and floors.

For small strains, the proportional increase of the material's extent, i.e. that of the volume of the domain throughout which the skeleton extends is

$$e = e_{xx} + e_{yy} + e_{zz}. \tag{3}$$

Subtracting minus the hydrostatic pressure p from the diagonal elements of the stress tensor in eqn (1)—remember that positive τ_{ii} is a tension—gives the tensor of effective stress τ'_{ij}

$$\begin{bmatrix} \tau_{xx}+p & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \tau_{yy}+p & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \tau_{zz}+p \end{bmatrix}. \tag{4}$$

Assume that the strains are known. What are the stresses? Because the torque on an element of volume is zero, each shear stress τ'_{ij}, $i \neq j$, is the same as τ'_{ji}. Each is the corresponding shear strain e_{ij} multiplied by the shear modulus μ of the skeleton structure, thus

$$\mu e_{xy} = \tau'_{xy} = \tau'_{yx} = \mu e_{yx}$$
$$\mu e_{yz} = \tau'_{yz} = \tau'_{zy} = \mu e_{zy}$$
$$\mu e_{zx} = \tau'_{zx} = \tau'_{xz} = \mu e_{xz}. \tag{5}$$

The effective stresses are more complicated for two reasons. First, a change in pore pressure at constant skeleton strain causes a change in effective stress, the total stress minus the negative of the pore pressure. Second, because of the cross coupling described by Poisson's ratio, extending an otherwise unrestrained specimen in one direction produces, in most materials, a shrinkage at right angles to the extension.

Take these effects in order. Were the skeleton unrestrained, a pressure p would cause the extent, the volume of the domain throughout which the skeleton extends, to change by $-p\delta$, a shrinkage, where δ is the pore pressure compressibility of the material. Biot and Willis (1957) call δ the "unjacketed compressibility". If the solid phase has uniform properties, which it may not have, δ is the intrinsic compressibility of skeleton material.

To return the skeleton to its original extent would require that it be expanded by an isotropic effective tensile stress given by minus this volumetric strain divided by κ, the compressibility of the skeleton at constant pore pressure. This is the "jacketed compressibility" of Biot and Willis (1957). The jacket applies stress to the solid skeleton, and a vent tube extends into the sample to keep the pore pressure at zero. (In articular cartilage, the bearing material on the ends of animal bones, κ is much greater than δ.) The final result is an isotropic effective stress

$$\tau'_{xx\,\text{pore pressure}} = \tau'_{yy\,\text{pore pressure}} = \tau'_{zz\,\text{pore pressure}} = p\delta/\kappa. \tag{6}$$

Now consider the effect of strains. Any combination of extensions can be described as the sum of equal extensions, positive or negative, along all three axes, an extension and an equal shortening along two axes at right angles to each other—call them east–west and north–south—and a positive or negative extension along the axis at right angles to these two. Let this vertical axis be the x axis.

The equal extensions along all axes cause a proportional increase $e_{\text{isotropic}}$ in the volume that is three times the extension in any one direction. If the pore pressure is unchanged, the isotropic tension that results is

$$\tau'_{ii\,\text{isotropic strain}} = (1/\kappa)e_{\text{isotropic strain}}. \tag{7}$$

Equal east–west extension and north–south compression cause neither strain nor stress along the x (vertical) axis, and need not be considered further.

The extension along the x (vertical) axis causes a tensile stress given by this strain times the confined compression modulus $\Xi = 2\mu + \lambda$ of the solid structure, therefore

$$\tau'_{xx\,\text{uniaxial strain}} = \Xi e_{xx\,\text{uniaxial strain}}. \tag{8}$$

[The confined compression modulus is like the Young's modulus except that lateral shrinkage or expansion of the sample is prevented. Kenyon (1976) noted that it appeared in three-dimensional consolidation theory.]

Adding together the separate constituents gives the effective stress in the x direction

$$\begin{aligned}\tau'_{xx} &= \tau'_{xx\,\text{pore pressure}} + \tau'_{xx\,\text{isotropic strain}} + \tau'_{xx\,\text{uniaxial strain}}\\ &= p\delta/\kappa + (1/\kappa)e_{\text{isotropic strain}} + \Xi e_{xx\,\text{uniaxial strain}}.\end{aligned} \tag{9}$$

The total extension along the x direction is

$$e_{xx} = (1/3)e_{\text{isotropic strain}} + e_{xx\,\text{uniaxial strain}} \tag{10}$$

and the total volumetric increase is

$$e = e_{\text{isotropic strain}} + e_{xx\,\text{uniaxial strain}}. \tag{11}$$

Subtracting eqn (10) from eqn (11) gives

$$e_{\text{isotropic strain}} + \ = (3/2)(e - e_{xx}) \tag{12}$$

and subtracting eqn (11) from three times eqn (10) gives

$$e_{xx\,\text{uniaxial}} = (3e_{xx} - e)/2. \tag{13}$$

Substituting eqns (12) and (13) into eqn (9) gives

$$\begin{aligned} \tau'_{xx} &= p\delta/\kappa + (3/2\kappa)(e - e_{xx}) + \Xi(3e_{xx} - e)/2 = p\delta/\kappa + 3e_{xx}[\Xi - (1/\kappa)]/2 + e(3/\kappa - \Xi)/2 \\ &= p\delta/\kappa + 2\mu e_{xx} + \lambda e \end{aligned} \tag{14}$$

where $\mu = (3/4)[\Xi - (1/\kappa)]$ is the shear modulus and $\lambda = [(3/\kappa) - \Xi]/2$ the Lamé coefficient of the skeleton, which is the same as that of the entire material at zero or constant pore pressure. Repeating the derivation for the y and z directions gives

$$\begin{aligned} \tau'_{xx} &= p\delta/\kappa + 2\mu e_{xx} + \lambda e \\ \tau'_{yy} &= p\delta/\kappa + 2\mu e_{yy} + \lambda e \\ \tau'_{zz} &= p\delta/\kappa + 2\mu e_{zz} + \lambda e. \end{aligned} \tag{15}$$

Equation (6) can now be made more general. If $e_{xx} + e_{yy} + e_{zz} = e = 0$, $\tau'_{xx} + \tau'_{yy} + \tau'_{zz} = 3p\delta/\kappa$, so

$$(\partial \tau'/\partial p)_{e=\text{const.}} = \delta/\kappa, \tag{16}$$

where $\tau' = (\tau'_{xx} + \tau'_{yy} + \tau'_{zz})/3$.

In conjunction with eqns (5), eqns (15) give the full specification of effective stresses in terms of strains. Equations (5) and (15) form eqn (43) in Biot and Willis (1957), because δ/κ equals Biot and Willis's $1 - \alpha$. If the skeleton material is intrinsically incompressible, $\delta = 0$ and the pressure drops out of eqns (15).

Since inertial reactions are assumed to be minute, the net force on every elemental volume of material must be effectively zero, so

$$\begin{aligned} (\partial/\partial x)(\tau'_{xx} - p) + \partial \tau'_{xy}/\partial y + \partial \tau'_{xz}/\partial z &= 0 \\ \partial \tau'_{yx}/\partial x + (\partial/\partial y)(\tau'_{yy} - p) + \partial \tau'_{yz}/\partial z &= 0 \\ \partial \tau'_{zx}/\partial x + \partial \tau'_{zy}/\partial y + (\partial/\partial z)(\tau'_{zz} - p) &= 0, \end{aligned} \tag{17}$$

which describe equilibrium in the x, y and z directions. This is eqn (2.10) in Biot (1955), but written in the stress superposition system and with body forces set equal to zero.

The terms on the main diagonal account for the difference in the total stress on opposing faces of an elemental cube and the off-diagonal terms account for the difference between shear stresses on opposing faces.

Substituting eqns (5) and (15) into eqns (17) gives

$$
\begin{aligned}
(\partial/\partial x)[2\mu e_{xx}+\lambda e-p(1-\delta/\kappa)]+\mu(\partial e_{xy}/\partial y)+\mu(\partial e_{xz}/\partial z) &= 0\\
\mu(\partial e_{yx}/\partial x)+(\partial/\partial y)[2\mu e_{yy}+\lambda e-p(1-\delta/\kappa)]+(\mu\,\partial e_{yz}/\partial z) &= 0\\
\mu(\partial e_{zx}/\partial x)+\mu(\partial e_{zy}/\partial y)+(\partial/\partial z)[2\mu e_{zz}+\lambda e-p(1-\delta/\kappa)] &= 0 \qquad (18)
\end{aligned}
$$

which describe equilibrium in the x, y and z directions. Regard this as a single vector equation and take its divergence.

$$
\begin{aligned}
&(\partial^2/\delta x^2)[2\mu e_{xx}+\lambda e-p(1-\delta/\kappa)]+\mu(\partial^2 e_{xy}/\partial x\,\partial y)+\mu(\partial^2 e_{xz}/\partial x\,\partial z)\\
&\quad+\mu(\partial^2 e_{yx}/\partial y\,\partial x)+(\partial^2/\delta y^2)[2\mu e_{yy}+\lambda e-p(1-\delta/\kappa)]+\mu(\partial^2 e_{yz}/\partial y\,\partial z)\\
&\quad+\mu(\partial^2 e_{zx}/\partial z\,\partial x)+\mu(\partial^2 e_{zy}/\partial z\,\partial y)+(\partial^2/\delta z^2)[2\mu e_{zz}+\lambda e-p(1-\delta/\kappa)]=0. \qquad (19)
\end{aligned}
$$

This is not as bad as it looks, because each mixed derivative of shear strain can be replaced by the sum of the second derivatives of extension in the two dimensions of the mixed derivative. From eqn (25) on p. 49 of Love (1944),

$$
\begin{aligned}
(\partial^2 e_{yz}/\partial y\,\partial z) &= (\partial^2 e_{yy}/\partial z^2)+(\partial^2 e_{zz}/\partial y^2) = (\partial^2 e_{zy}/\partial z\,\partial y)\\
(\partial^2 e_{zx}/\partial z\,\partial x) &= (\partial^2 e_{zz}/\partial x^2)+(\partial^2 e_{xx}/\partial z^2) = (\partial^2 e_{xz}/\partial x\,\partial z)\\
(\partial^2 e_{xy}/\partial x\,\partial y) &= (\partial^2 e_{xx}/\partial y^2)+(\partial^2 e_{yy}/\partial x^2) = (\partial^2 e_{yx}/\partial y\,\partial x). \qquad (20)
\end{aligned}
$$

(The left hand equalities are given by Love. The right hand equalities follow from the left because of the symmetry of the expressions between the equals signs.) Love derived these relations by expressing the strains as space derivatives of the displacement of material points in the sample. Biot, in his derivation, expresses strains this way (without saying why he does so), and thus implicitly uses eqns (20). Substituting eqns (20) into eqn (19) gives

$$
\begin{aligned}
&(\partial^2/\delta x^2)[2\mu e_{xx}+\lambda e-p(1-\delta/\kappa)]+\mu[(\partial^2 e_{xx}/\partial y^2)+(\partial^2 e_{yy}/\partial x^2)\\
&\quad+(\partial^2 e_{xx}/\partial z^2)+(\partial^2 e_{zz}/\partial x^2)]+\mu[(\partial^2 e_{xx}/\partial y^2)+(\partial^2 e_{yy}/\partial x^2)]+(\partial^2/\delta y^2)[2\mu e_{yy}\\
&\quad+\lambda e-p(1-\delta/\kappa)]+\mu[(\partial^2 e_{zz}/\partial y^2)+(\partial^2 e_{yy}/\partial z^2)]+\mu[(\partial^2 e_{xx}/\partial z^2)\\
&\quad+(\partial^2 e_{zz}/\partial x^2)+(\partial^2 e_{zz}/\partial y^2)+(\partial^2 e_{yy}/\partial z^2)]+(\partial^2/\delta z^2)[2\mu e_{zz}+\lambda e-p(1-\delta/\kappa)]\\
&=(\partial^2/\delta x^2)[2\mu(e_{xx}+e_{yy}+e_{zz})+\lambda e-p(1-\delta/\kappa)]+(\partial^2/\delta y^2)[2\mu(e_{xx}+e_{yy}+e_{zz})\\
&\quad+\lambda e-p(1-\delta/\kappa)]+(\partial^2/\delta z^2)[2\mu(e_{xx}+e_{yy}+e_{zz})[\lambda e-p(1-\delta/\kappa)]\\
&=\nabla^2[6\mu e+3\lambda e-3p(1-\delta/\kappa)] = 3\nabla^2[\Xi e-p(1-\delta/\kappa)]=0,
\end{aligned}
$$

or

$$
\nabla^2 e = \nabla^2 p(1-\delta/\kappa)/\Xi. \qquad (21)
$$

The Laplacian of skeleton expansion is proportional to the Laplacian of pressure. For the special case when $\delta = 0$, and provided that porosity has a definite meaning, this is eqn (4.15) of Biot (1955). Biot (1955) does not give the general case.

This relation results from the equilibrium of forces, introduced as eqn (17). The force on an elemental volume of solid skeleton due to a gradient of effective stress is balanced by the force due to a gradient of pore pressure.

Equation (21) contains neither time nor the compressibility of the fluid. It relates dilatation to pressure, but says nothing about how these change with time as the fluid flows through the material. The consequences of fluid flow will now be considered.

ENTER FLOW

Darcy's law says that fluid flows into a unit volume of material at a rate $(1/K)\nabla^2 p$ where K is the flow resistivity. This fluid will expand the material, increase the pressure in its pores, or both, so

$$[\Xi/(1-\delta/\kappa)]\nabla^2 e = \nabla^2 p = K[(\partial\zeta/\partial e)_{p=\text{const.}}\,\partial e/\partial t + (\partial\zeta/\partial p)_{e=\text{const.}}\,\partial p/\partial t], \tag{22}$$

where ζ is the volume of fluid in a unit volume of the material (terminology of Biot and Willis, 1957). This is one form of the consolidation equation. Its first equality is eqn (21).

Porosity does not appear explicitly. It falls as material is compressed, but the fall affects consolidation only through lowered permeability and increased stiffness.

INVOKE THE STRAIN ENERGY OF THE SOLID

Since Ξ, $\delta/\kappa = (\partial\tau'/\partial p)_{e=\text{const.}}$, $(\partial\zeta/\partial e)_{p=\text{const.}}$, $(\partial\zeta/\partial p)_{e=\text{const.}}$ and K are measurable material constants, eqn (22), in principle, solves the problem. Measure them, insert the initial and boundary conditions, and eqn (22) predicts what the material will do. But because the solid is assumed to be elastic $(\partial\zeta/\partial e)_{p=\text{const.}} = 1-\delta/\kappa$, as shown below, which eliminates one coefficient.

From eqn (16), $(\partial\tau'/\partial p)_{e=\text{const.}} = \delta/\kappa$. Subtract both sides of this from unity to get

$$[\partial(p-\tau')/\partial p]_{e=\text{const.}} = 1-\delta/\kappa. \tag{23}$$

Because the solid is assumed to be elastic its strain energy W is a single-valued function of e and ζ. Therefore

$$\begin{aligned}\text{Curl Grad } W &= [\partial(\partial W/\partial\zeta)_{e=\text{const.}}/\partial e]_{\zeta=\text{const.}} - [\partial(\partial W/\partial e)_{\zeta=\text{const.}}/\partial\zeta]_{e=\text{const.}}\\ &= (\partial p/\partial e)_{\zeta=\text{const.}} - [\partial(\tau'-p)/\partial\zeta]_{e=\text{const.}} = 0.\end{aligned} \tag{24}$$

So

$$(\partial\zeta/\partial e)_{p=\text{const.}} = 1/(\partial e/\partial\zeta)_{p=\text{const.}} = [\partial(\tau'-p)/\partial\zeta]_{e=\text{const.}}(\partial e/\partial p)_{\zeta=\text{const.}}/(\partial e/\partial\zeta)_{p=\text{const.}}$$

(because, from eqn (24) $[\partial(\tau'-p)/\partial\zeta]_{e=\text{const.}}/(\partial p/\partial e)_{\zeta=\text{const.}} = 1$) $= -[\partial(\tau'-p)/\partial\zeta]_{e=\text{const.}}$ $(\partial\zeta/\partial p)_{e=\text{const.}}$ [from the cyclic rule for partial derivatives (Gellert *et al.* 1977)]

$$= -[\partial(\tau'-p)/\partial p]_{e=\text{const.}} = 1-\delta/\kappa, \tag{25}$$

from eqn (23), and eqn (22) becomes

$$[\Xi/(1-\delta/\kappa)]\nabla^2 e = \nabla^2 p = K[(1-\delta/\kappa)\,\partial e/\partial t + (\partial\zeta/\partial p)_{e=\text{const.}}\,\partial p/\partial t], \tag{26}$$

or, using Biot's $M = (\partial p/\partial\zeta)_{e=\text{const.}}$,

$$[\Xi/(1-\delta/\kappa)]\nabla^2 e = \nabla^2 p = K[(1-\delta/\kappa)\,\partial e/\partial t + (1/M)\,\partial p/\partial t]. \tag{27}$$

Equations (22), (26) and (27) are different versions of the tells-all, run-on equation of consolidation.

Compressibility of the fluid does not appear explicitly. It exerts its effect through the coefficient $1/M$. Any influence that nearby solid has on the fluid's compressibility will appear in the measured value of M.

The theory does not depend on the material's consisting of identifiable phases with uniform properties. If the phases are uniform, the pore pressure compressibility δ of the domain occupied by the skeleton is the compressibility of bulk solid.

If the solid is very stiff, Ξ is very large, and the left hand equality in eqn (27) shows that e becomes very small. So, therefore, does $\partial e/\partial t$. Alternatively, if the fluid is very compressible, M is very small so the $\partial p/\partial t$ term becomes very large. In either case the $\partial e/\partial t$ term is much smaller than the $\partial p/\partial t$ term and eqn (27) approaches

$$[\Xi/(1-\delta/\kappa)]\nabla^2 e = \nabla^2 p = K(1/M)\,\partial p/\partial t, \tag{28}$$

the right half of which is a diffusion equation in pressure, the equation for a gas percolating through a rigid sponge. The speed of percolation is proportional to M, the rise in pressure when fluid is forced into the material, and inversely proportional to the flow resistivity K.

If the constituents of the material are incompressible, $(\partial\zeta/\partial p)_{e=\text{const.}} = 1/M = \delta = 0$, and eqn (27) reduces to

$$\nabla^2 p = \Xi\nabla^2 e = K\,\partial e/\partial t, \tag{29}$$

the right-hand half of which is a diffusion equation for the volumetric strain. It describes the change in deformation of an elastic sponge made of intrinsically incompressible material like an idealized rubber soaked with incompressible fluid, and is a good approximate equation for articular cartilage. The speed of the change is proportional to the confined compression modulus Ξ of the material at constant pressure and inversely proportional to K. In this approximation only Ξ and K need be measured to predict what the material will do.

If the porosity can be defined the right-hand equality in eqn (29) corresponds to eqn (4.16) in Biot (1955).

If neither the $(1-\delta/\kappa)\,\partial e/\partial t = (\partial\zeta/\partial e)_{p=\text{const.}}\,\partial e/\partial t$ term nor the $(1/M)\,\partial p/\partial t = (\partial\zeta/\partial p)_{e=\text{const.}}\,\partial p/\partial t$ term can be ignored, eqn (27) is not a diffusion equation unless $\partial e/\partial t$ and $\partial p/\partial t$ are everywhere in the same proportion to each other and remain so. In three dimensions this in general does not happen.

Consider a cylindrical specimen immersed in a bath of pore fluid at zero pressure. Put it under rapidly increasing tension. In its interior some of the tension will be borne by negative pressure in its pore fluid. At its surface none will be so borne because the pore pressure will be zero to match that in the bath. The ratio of $|\partial p/\partial t|$ to $|\partial e/\partial t|$ is much higher deep in the interior than near the surface.

IN ONE DIMENSION

If the specimen is infinitely broad the situation becomes one-dimensional. If this dimension lies in the x direction, eqn (27) becomes

$$[\Xi/(1-\delta/\kappa)]\,\partial^2 e_{xx}/\partial x^2 = \partial^2 p/\partial x^2 = K[(1-\delta/\kappa)\,\partial e_{xx}/\partial t + (1/M)\partial p/\partial t]. \tag{30}$$

Equation (17) for the equilibrium of forces becomes

$$(\partial/\partial x)(\tau'_{xx} - p) = 0.$$

Integrating this with respect to x gives $\tau'_{xx} - p = \text{const.} = \tau_{0xx}(t)$, so

$$\tau'_{xx} = p + \tau_{0xx}(t), \tag{31}$$

where $\tau_{0xx}(t) = \tau'_{xx} - p$ is the total tensile stress on the sample. It is constant throughout the depth of the sample, but may vary with time. Substituting eqn (31) into the first of eqns (15)

$$\tau'_{xx} = p + \tau_{0xx}(t) = p\delta/\kappa + (2\mu+\lambda)e_{xx} = p\delta/\kappa + \Xi e_{xx},$$

so

$$e_{xx} = [p(1-\delta/\kappa)+\tau_{0xx}(t)]/\Xi, \tag{32}$$

from which

$$\partial e_{xx}/\partial t = \{(1-\delta/\kappa)\partial p/\partial t+\mathrm{d}[\tau_{0xx}(t)]/\,\mathrm{d}t\}/\Xi. \tag{33}$$

If $\mathrm{d}[\tau_{0xx}(t)]/\mathrm{d}t$ is zero, $\partial e_{xx}/\partial t$ is in constant proportion to $\partial p/\partial t$ and either e_{xx} or p can be eliminated from eqn (30) to give a diffusion equation in p or e_{xx}. So long as the loading is constant, eqn (30) is a diffusion equation. This is true whatever the boundary conditions are in other respects. The force may be applied by impermeable anvils, permeable anvils or by fluid alone at either end in any combination, and these conditions may change during the period of observation.

COMPARISON WITH BIOT

For cases where the porosity has a definite meaning, the right hand equality in eqn (27) is the end-to-end equality in Biot's run-on eqn (4.9) (Biot, 1955). Biot does not give the end-to-end equality in eqn (27) because it depends on eqn (21), for which Biot (1955) does not give the general case. Variables and coefficients correspond thus, McCutchen to the left of the first equals sign, Biot (1955) and Biot and Willis (1957) to the right:

$$\text{Pressure}\, p = -\sigma/f,$$

where f is the porosity, the volume of the pores divided by the total volume.

$$\text{Flow resistivity}\ K = b/f^2,$$

$$1-\delta/\kappa = \alpha = f(Q+R)/R,$$

$$M = M = R/f^2.$$

Q and R are coefficients in Biot's expression for strain energy. By making the above substitutions, Biot could have eliminated porosity from his eqn (4.9).

Biot and Willis did not treat M as a quantity one might measure. Instead the measured quantity would be $\gamma = (\partial\zeta/\partial p)_{\tau'=\mathrm{const.}}$, the coefficient of fluid content, the amount of fluid that enters a unit volume of unrestrained sample when unit pressure is applied. The boundary across which this flux is defined shrinks with the sample as the pressure rises.

To relate M to γ, first raise the pressure on a sample by Δp. The extent of the sample will shrink by $\delta\Delta p$, and its fluid content ζ will rise by $\gamma\Delta p$. Now stretch it back to its original extent at constant pressure. Because by eqn (25) $(\partial\zeta/\partial e)_{p=\mathrm{const.}} = 1-\delta/\kappa$, its fluid content will rise by a further $\delta\Delta p(\partial\zeta/\partial e)_{p=\mathrm{const.}} = \delta\Delta p(1-\delta/\kappa)$, making ζ increase in total by $\Delta p[\gamma+\delta(1-\delta/\kappa)]$. So

$$(\partial p/\partial\zeta)_{e=\mathrm{const.}} = M = 1/[\gamma+\delta(1-\delta/\kappa)] = R/f^2, \tag{34}$$

the last equality from Biot and Willis (1957)'s eqn (21). Equation (34) can also be derived by eliminating e between eqns (36) and (37) below and differentiating the result to get $(\partial\zeta/\partial p)_{\tau'=\mathrm{const.}} = \gamma$.

ALTERNATIVELY, FROM AN EXPLICIT EXPRESSION FOR STRAIN ENERGY

Readers who distrust the physical reasoning used to derive various quantities may prefer to see them derived from strain energy. Since the solid is assumed to be Hookean, its potential energy W is

$$W=(1/2)(Ze^2+2Be\zeta+M\zeta^2) \tag{35}$$

per unit volume where e and ζ are taken to be zero when p and $\tau'=(\tau'_{xx}+\tau'_{yy}+\tau'_{zz})/3$ are zero. (Up to here the zeros of stresses and strains have not mattered.) M is the quantity used in Biot and Willis (1957). The meaning of Z and B will soon be evident. Shear strains are not considered because they affect neither the volume of a sample nor, if the material is linear, its effective tensile or compressive stresses. (See eqn (15) or eqn (4.3) of Biot, 1955.)

Equation (35) resembles eqn (4.1) of Biot, 1955, but it considers the energy as a function of e, the extent of the solid and ζ, the fluid content, rather than e and the dilatation ε of the fluid.

Change in internal energy of the material equals work done upon it. Since the increments of work are $(\tau'-p)\,\mathrm{d}e$ and $p\,\mathrm{d}\zeta$, the former because $\tau'-p$ is the tensile total stress upon the surface of the sample,

$$\tau'-p=(\partial W/\partial e)_{\zeta=\text{const.}}=Ze+B\zeta \tag{36}$$

$$p=(\partial W/\partial\zeta)_{e=\text{const.}}=Be+M\zeta. \tag{37}$$

Differentiate eqn (36) with respect to ζ at constant e to get $B=(\partial(\tau'-p)/\partial\zeta)_{e=\text{const.}}$, the change in total tensile stress $\tau'-p$ consequent on an injection of fluid at constant extent, and differentiate eqn (37) with respect to e at constant ζ to get $B=(\partial p/\partial e)_{\zeta=\text{const.}}$, the change in pore pressure occasioned by a change in extent in the absence of gain or loss of fluid. B has two meanings because it is the coefficient of the cross term in eqn (35). The resulting equality $(\partial(\tau'-p)/\partial\zeta)_{e=\text{const.}}=(\partial p/\partial e)_{\zeta=\text{const.}}$ is eqn (24), derived via Curl Grad $W=0$.

Differentiate eqn (37) with respect to ζ at constant e, getting

$$(\partial p/\partial\zeta)_{e=\text{const.}}=M, \tag{38}$$

so

$$\begin{aligned}B/M&=[\partial(\tau'-p)/\partial p]_{e=\text{const.}}=\partial\tau'/\partial p)_{e=\text{const.}}-1\\&=-(\partial e/\partial p)_{\tau'=\text{const.}}/(\partial e/\partial\tau')_{p=\text{const.}}-1\text{ [from the cyclic rule (Gellert et al., 1977)]}\\&=\delta/\kappa-1\end{aligned} \tag{39}$$

from which

$$(\partial\tau'/\partial p)_{e=\text{const.}}=\delta/\kappa,\text{ which is eqn 16, and}$$
$$B=-(1-\delta/\kappa)M. \tag{40}$$

Equation (40) can be used to write the consolidation eqn (26) in terms of the coefficients in the energy expression eqn (35) thus:

$$-(\Xi M/B)\nabla^2 e=\nabla^2 p=K[-(B/M)\,\partial e/\partial t+(1/M)\,\partial p/\partial t]. \tag{41}$$

From eqn (36) $Z=[\partial(\tau'-p)/\partial e]_{\zeta=\text{const.}}$ is the tensile total stress $\tau'-p$ produced by extending the material while not allowing fluid to enter it. It is the bulk modulus of the material in the absence of redistribution of fluid, and is very close to what one would measure even with flow permitted if the experiment were done very quickly.

Coefficient Z does not appear in eqn (41). Alone of the coefficients M, B, and Z in the potential energy eqn (35), Z is not multiplied by ζ, and concerns only situations in which fluid does not move relative to the solid, so there is no consolidation. For consolidation theory the confined compression modulus Ξ and any two of M, B and $\delta/\kappa=(\partial\tau'/\partial p)_{e=\text{const.}}$

are sufficient. δ and κ do not need to be known independently. Z can replace M or B, but then κ must be known as well as δ/κ to find M or B.

To express Z in terms of M and other constants first expand a sample volumetrically by Δe at constant pore pressure. This will cause an isotropic effective tensile stress $\Delta e/\kappa$ in the skeleton, and by eqn (25) a volume $\Delta e(1-\delta/\kappa)$ of fluid will enter it. Now remove this fluid at constant e. The pressure will drop by $M\Delta e(1-\delta/\kappa)$, and the effective tensile stress will drop by $M\Delta e\delta/\kappa(1-\delta/\kappa)$ by eqn (16). The change in total stress $\Delta eZ = \Delta e\{1/\kappa+M[(1-\delta/\kappa)-\delta/\kappa(1-\delta/\kappa)]\} = \Delta e[1/\kappa+M(1-\delta/\kappa)^2]$, so, because $B = -(1-\delta/\kappa)M$

$$Z = 1/\kappa+M(1-\delta/\kappa)^2 = 1/\kappa+B^2/M = 1/\kappa-B(1-\delta/\kappa). \tag{42}$$

Alternatively, eliminate ζ between eqns (36) and (37), differentiate the result to get $(\partial\tau'/\partial e)_{p=\mathrm{const.}} = 1/\kappa$ and solve the resulting equation for Z.

GOOD AND BAD COEFFICIENTS

The effective tensile stress of a material as a function of extent and pore pressure can be expressed in terms of any three independent coefficients that can be extracted from eqns (36) and (37). Though M, B and Z are themselves measurable, they all involve compressing solid or fluid rather than merely deforming the structure of the solid. Coefficient κ is always larger than δ and $1/M$. When, as in cartilage, the solid structure is easily deformed κ is very much larger, so $1-\delta/\kappa$ is little different from unity. M, B and Z are then of about the same magnitude and a bad set to measure.

The effective stress τ' depends on the differences between the magnitudes of M, B and Z, as shown by adding eqns (36) and (37) to get

$$\tau' = (Z+B)e+(B+M)\zeta. \tag{43}$$

(Remember that M and Z are positive and B negative.) Using eqns (40) and (42) to express Z and B in terms of M, κ and $1-\delta/\kappa$ gives these differences directly.

$$\tau' = [1/\kappa+M(1-\delta/\kappa)^2-(1-\delta/\kappa)M]e+[M-(1-\delta/\kappa)M]\zeta,$$

in which M and $-M$ twice cancel to give

$$\tau' = \{1/\kappa-[\delta/\kappa-(\delta/\kappa)^2]M\}e+(\delta/\kappa)M\zeta, \tag{44}$$

a stress much smaller than Me or $M\zeta$ if κ is much larger than δ and $1/M$.

Likewise, the strain energy, minus any contribution from shears, can be written

$$W = (1/2)\{[1/\kappa+M(1-\delta/\kappa)^2]e^2-2(1-\delta/\kappa)Me\zeta+M\zeta^2\}. \tag{45}$$

EXPRESS POTENTIAL ENERGY IN TERMS OF STRESSES, NOT DISPLACEMENTS

M, δ, and κ are good coefficients, but eqn (45) can be a poor equation. In cartilage κ is much larger than δ and $1/M$, e approximately equals ζ and the cross term in eqn (45) nearly cancels the other two. Most of the strain energy lies in the deformation of the solid skeleton rather than compression of the materials. This is well concealed in eqn (45), but obvious if strain energy is written in terms of τ' and p rather than e and ζ. To do this note first that eqn (35) can be written as the sum of two squares, and that via eqns (42) and (37)

$$\begin{aligned} W &= (1/2)[(Z-B^2/M)e^2+(B^2e^2+2BeM\zeta+M^2\zeta^2)/M] \\ &= (1/2)[(Z-B^2/M)e^2+(Be+M\zeta)^2/M] \\ &= (1/2)[(e^2/\kappa)+p^2/M] = (1/2)[e^2(\partial\tau'/\partial e)_{p=\text{const.}}+p^2(\partial\zeta'/\partial p)_{e=\text{const.}}]. \end{aligned} \tag{46}$$

Here the energy is written as a function of extent e and pressure p. There is no cross term.

(There is no cross term because an extension e at zero p causes a rise in potential energy proportional to e^2. Because the pressure remains zero a subsequent change in p causes a rise in energy proportional to p^2. An extension e at zero ζ, on the other hand, causes a rise in energy proportional to e^2, but it also changes the pressure p so a subsequent change in ζ does not cause a rise in energy proportional to ζ^2, hence the presence of a cross term in eqns (35) and (45).

For completeness note that eqn (35) can be expressed as the sum of a different pair of squares. Coupled with eqns (36) and (42) this gives

$$\begin{aligned} W &= (1/2)[(Z^2e^2+2ZBe\zeta+B^2\zeta^2)/Z+(M-B^2/Z)\zeta^2] \\ &= (1/2)[(Ze+B\zeta)^2/Z+(M-B^2/Z)\zeta^2] \\ &= (1/2)\{(\tau'-p)^2/Z+[M/(\kappa Z)]\zeta^2\} \\ &= (1/2)\{(\tau'-p)^2[\partial e/\partial(\tau'-p)]_{\zeta=\text{const.}}+\zeta^2(\partial p/\partial\zeta)_{\tau'-p=\text{const.}}\} \end{aligned} \tag{47}$$

in total stress $\tau'-p$ and fluid content ζ, again with no cross term.)

Eliminate ζ between eqns (36) and (37), use eqns (40) and (42) getting

$$e = \kappa\tau'-\delta p \tag{48}$$

and substitute this in eqn (46) to get

$$W = (1/2)\{\kappa\tau'^2-2\delta\tau' p+[(\delta^2/\kappa)+1/M]p^2\}. \tag{49}$$

In consolidation the gradients of pressure and effective stress balance each other, so τ' and p are of the same order. In cartilage the intrinsic compressibility δ of the solid and $1/M$ are of the same order. The effective-stress compressibility κ of the solid structure is much larger, so $\delta/\kappa \ll 1$. Thus, in eqn (49) $(\delta^2/\kappa) \ll 1/M$, and roughly

$$W = (1/2)(\kappa\tau'^2-2\delta\tau' p+p^2/M), \tag{50}$$

in which the first term is much the largest. Most of the strain energy is in deformation of the skeleton structure.

If all the constituents of the material are incompressible $\delta = 0 = 1/M$, so

$$W = (1/2)\kappa\tau'^2 = (1/2)e^2/\kappa = (1/2)e^2(\Xi/3)(1+\nu)/(1-\nu), \tag{51}$$

where ν is the Poisson's ratio at constant pore pressure.

ARTICULAR CARTILAGE

Estimate Z, B and M for cartilage. Take the compressibility $1/Z$ of the whole material to be the same as the pore pressure compressibility δ of the solid. Pore fluid must then have the same compressibility. When the whole material is pressurized everything shrinks to the same degree and ζ is zero. Equation (42) then becomes

$$Z = 1/\kappa - B(1-\delta/\kappa) = 1/\kappa - B(1-1/Z\kappa), \text{ so}$$

$$B = (1/\kappa - Z)/(1-1/Z\kappa) = -Z. \tag{52}$$

Assume δ to be 4.44×10^{-5} cm^2/kg, the compressibility of water. Then $Z = 1/\delta = 2.25 \times 10^4$ kg/cm^2. For Poisson's own proposed Poisson's ratio, 1/4 (Love, 1944, p. 13), the bulk modulus $1/\kappa$ of the structure is 5/9 the confined compression modulus, about 5.8 kg/cm^2 (McCutchen, 1962, wrongly called it the Young's modulus. Though unconfined laterally, the sample was much thinner in the direction of compression than across it, and compressed between rough-surfaced anvils that must have prevented its surfaces from slipping. The figure is for compressive deformation. Extension tautens the collagen fibers in the cartilage which makes the modulus much higher.) The bulk modulus of the cartilage structure is therefore about 4 kg/cm^2, its compressibility 0.25 cm^2/kg, $\delta/\kappa = 1.776 \times 10^{-4}$, $1-\delta/\kappa = 0.9998224$ and $M = 1.000177Z = 2.250399 \times 10^4$ kg/cm^2. It is because $\kappa \gg \delta$ that Z, M and B are all of about the same magnitude.

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